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## Singularities of the Bergman kernel for certain weakly pseudoconvex domains

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Let  $\Omega$  be a bounded domain with smooth boundary in  $\mathbb{C}^n$ ,  $B(\Omega)$  the set of holomorphic  $L^2$ -functions on  $\Omega$ . It is well-known that  $B(\Omega)$  is a closed linear subspace of the Hilbert space  $L^2(\Omega)$ . The *Bergman kernel*  $K^B(z)$  of the domain  $\Omega$  is defined by

$$K^B(z) = \sum_j |\phi_j(z)|^2,$$

where  $\{\phi_j\}$  is a complete orthonormal basis for  $B(\Omega)$ . The above series converges uniformly on any compact subset of  $\Omega$ . It is very important to investigate the singularities of  $K^B(z)$ . This is mainly because they contain much information about the analytic and geometric invariants of the domain  $\Omega$ .

First we consider the case where  $\Omega$  is a strongly pseudoconvex domain. In this case C. Fefferman [13], L. Boutet de Monvel and J. Sjöstrand [5] obtained the following asymptotic expansion for  $K^B(z)$ :

$$K^B(z) = \frac{\varphi^B(z)}{r(z)^{n+1}} + \psi^B(z) \log r(z), \quad (1)$$

where  $r$  is a defining function of  $\Omega$ , i.e.,  $\Omega = \{z \in \mathbb{C}^n; r(z) > 0\}$  and  $\text{grad} r(z) \neq 0$  on  $\partial\Omega$ . The functions  $\varphi^B(z)$  and  $\psi^B(z)$  can be expressed as a power series of  $r$ . If the boundary  $\partial\Omega$  is real analytic, then their series are convergent. From the viewpoint of ordinary differential equations, this result may be interpreted that the Bergman kernel of a strongly pseudoconvex domain has the singularities of *regular singular type*.

Next we proceed to the case of weakly pseudoconvex domain of finite type (in the sense of J. J. Kohn [24] or J. P. D'Angelo [9]). In this case there is no such strong general result

that is comparable with (1) in the strongly pseudoconvex case ; yet there are many detailed results for specific domains. We refer to [2],[8],[16],[10],[14],[17] for explicit computations, to [18],[35],[12],[6],[19],[20] for estimates of the size and to [3] for boundary limits on non tangential cone. Especially D. Catlin [6] and G. Herbort [20] gave precise estimates of  $K^B(z)$  from above and below for certain class of domains whose degenerate rank of the Levi form equals one. In general, however, the singularities of  $K^B(z)$  are so complicated that a unified treatment of them seems to be difficult.

In this article, we pick up the specific domains

$$\mathcal{E}_m = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n ; \sum_{j=1}^n |z_j|^{2m_j} < 1 \right\}, \quad (2)$$

where  $m = (m_1, \dots, m_n) \in \mathbb{N}^n$  and  $m_n \neq 1$ , to clarify what is happening for the weakly pseudoconvex domains of finite type. Since  $\mathcal{E}_m$  is a Reinhardt domain, the set of (normalized) monomials forms a complete orthonormal basis for  $B(\mathcal{E}_m)$ . Hence  $K^B(z)$  can be represented by a convergent power series of  $(|z_1|^2, \dots, |z_n|^2)$ , whose coefficients were explicitly computed in [21],[8],[4]. A. Bonami and N. Lohoué [4] gave an important integral representation for the Bergman kernel  $K^B(z)$  of  $\mathcal{E}_m$ . From this representation they deduced a detailed information about the singularities of  $K^B(z)$ , though their result is yet to be improved.

From our point of view, we briefly review the result of [4]. Let  $z^0 = (z_1^0, \dots, z_n^0) \in \partial\mathcal{E}_m$  be any boundary point of  $\mathcal{E}_m$ ,  $k \in \mathbb{Z}_{\geq 0}$  the degenerate rank of the Levi form at  $z^0$ . We say that  $z^0$  is a *strongly* (resp. *weakly*) *pseudoconvex point* if  $k = 0$  (resp. if  $k > 0$ ). Let  $I, P$  and  $Q$  be the subsets of  $N = \{1, \dots, n\}$  defined by

$$\begin{cases} I &= \{j \in N; m_j = 1\}, \\ P &= \{j \in N; z_j^0 = 0\} \setminus I, \\ Q &= \{j \in N; z_j^0 \neq 0\} \cup I. \end{cases}$$

Then the degenerate rank  $k$  equals the cardinality  $|P|$  of  $P$ . One of the main results in [4] (p.181) states that the restriction of  $K^B(z)$  to the subset  $V = \{z_j = 0; j \in P\}$  admits the following expression around  $z^0$ :

$$K^B(z) = C_P^B \frac{\prod_{j \in Q} m_j^2 |z_j|^{2m_j-2}}{\{1 - \sum_{j \in Q} |z_j|^{2m_j}\}^{|Q| + \frac{1}{m}|P|+1}} + O(1), \quad (3)$$

where  $C_P^B$  is a positive constant and  $|\frac{1}{m}|_P = \sum_{j \in P} \frac{1}{m_j}$ . The formula (3) is quite explicit, but still weak in the sense that it is valid only on the thin set  $V$ , and that the error term  $O(1)$  is somewhat too loose.

Besides [4] there are some studies on the Bergman kernel (or Szegő kernel) of the domain  $\mathcal{E}_m$  ([2],[21],[8],[15]). In the case  $m = (1, \dots, 1, m)$ , explicit expressions for  $K^B(z)$  are computed ([2],[8],[4]), while there seems to be no explicit one for general  $m$ . Recently, N. W. Gebelt [15] generalized the method of producing the asymptotic expansion (1) due to Fefferman [13] to the weakly pseudoconvex case of  $\mathcal{E}_m$  and obtained the analogous results about  $K^B(z)$  of  $\mathcal{E}_m$  ( $m = (1, \dots, 1, m)$ ).

Now we state our main results. Our essential idea is to introduce the new variables  $(t, r)$ , which we call the *polar coordinates* around  $z^0$ . Here  $t = (t_j)_{j \in P}$  is defined by

$$t_j(z)^{2m_j} = \frac{|z_j|^{2m_j}}{1 - \sum_{j \in Q} |z_j|^{2m_j}} \quad (j \in P),$$

and  $r$  is the defining function of  $\mathcal{E}_m$ , i.e.,

$$r(z) = 1 - \sum_{j=1}^n |z_j|^{2m_j}.$$

We call  $t$  the *angular variables* and  $r$  the *radial variable*, respectively. Then the map  $F : z \mapsto (t, r)$  takes  $\mathcal{E}_m$  onto the region:

$$D = \left\{ (t, r) \in \mathbf{R}^{|P|} \times (0, 1]; t_j \geq 0, \sum_{j \in P} t_j^{2m_j} \leq 1 - r \right\}.$$

The accumulation points of  $F(z)$  as  $\mathcal{E}_m \ni z \rightarrow z^0$  are precisely those points which belong to the set  $\{0\} \times \overline{\Delta}$ , where  $\overline{\Delta}$  is the closure of the *locally closed* simplex:

$$\Delta = \left\{ t = (t_j)_{j \in P}; t_j \geq 0, \sum_{j \in P} t_j^{2m_j} < 1 \right\}.$$

Let  $G = U \cap K$  be a locally closed subset of an Euclidean space, where  $U$  is open and  $K$  is closed, respectively. Then we say that  $f \in C^\omega(G)$  if  $f$  is a real analytic function on some open neighborhood  $V$  of  $G$  in  $U$ , where  $V$  may depend on  $f$ .

The following theorem asserts that the asymptotic behavior of  $K^B$  as  $\mathcal{E}_m \ni z \rightarrow z^0$  can be expressed most conveniently in terms of the polar coordinates  $(t, r)$ .

**THEOREM 1** *There is a function  $\Phi^B(t) \in C^\omega(\Delta)$  such that*

$$K^B(z) \equiv \frac{n!}{\pi^n} \prod_{j \in Q} m_j^2 |z_j|^{2m_j-2} \frac{\Phi^B(t(z))}{r(z)^{|Q|+|\frac{1}{m}|_P+1}} \quad \text{modulo } C^\omega(\{z^0\}). \quad (4)$$

Here  $\Phi^B(t)$  satisfies (i) or (ii).

- (i) *If  $z^0$  is a strongly pseudoconvex point ( i.e.  $P = \emptyset$ ),  $\Phi^B(t) = 1$  identically.*
- (ii) *If  $z^0$  is a weakly pseudoconvex point ( i.e.  $P \neq \emptyset$ ), then  $\Phi^B(t)$  is positive on  $\Delta$  and is unbounded as  $t \in \Delta$  approaches  $\overline{\Delta} \setminus \Delta$ .*

*Remark.* The function  $\Phi^B(t)$  is essentially the Laplace transform of a certain auxiliary function expressible in terms of Mittag-Leffler's function, i.e.

$$\Phi^B(t) = \frac{1}{n!} \left[ 1 - \sum_{j \in P} t_j^{2m_j} \right]^{|Q|+|\frac{1}{m}|_P+1} \int_0^\infty e^{-s} \prod_{j \in P} F_{m_j}(t_j^2 s^{\frac{1}{m_j}}) s^{|Q|+|\frac{1}{m}|_P} ds, \quad (5)$$

where

$$F_m(u) = m \sum_{\nu=0}^{\infty} \frac{u^\nu}{\Gamma(\frac{\nu}{m} + \frac{1}{m})}.$$

We mention a few implications of the formula (4) in order to compare it with the known results stated previously. First, if  $z^0$  is a strongly pseudoconvex point, i.e.  $P = \emptyset$ , then the angular variables  $t$  do not appear and  $\Phi^B(t) = 1$  identically, and therefore (4) reproduces the asymptotic expansion (1) due to C. Fefferman [13], L. Boutet de Monvel and J. Sjöstrand [5]. We remark that the logarithmic term in (1) does not appear in the present case. Secondly, the restriction of (4) to the subset  $V$  is just the substitution  $t(z) = 0$  into (4), which induces the formula (3) with the error term  $O(1)$  replaced by a real analytic function. Thus the formula (4) improves that of Bonami and Lohoué [4] in the sense that it is valid in a wider domain and that the error term is more accurate.

From the above theorem we consider the behavior of  $K^B(z)$  at a weakly pseudoconvex point from the following three angles: (a) *estimate*, (b) *boundary limit* and (c) *asymptotic formula*. We assume  $z^0$  is a weakly pseudoconvex point and define the region  $\mathcal{U}_\alpha(z^0)(=\mathcal{U}_\alpha)$  by

$$\mathcal{U}_\alpha = \left\{ z \in \mathcal{E}_m; \sum_{j \in P} t_j(z) = \frac{\sum_{j \in P} |z_j|^{2m_j}}{1 - \sum_{j \in Q} |z_j|^{2m_j}} < \frac{1}{\alpha} \right\} \quad (\alpha > 1).$$

(a) By the boundedness of  $\Phi^B(t)$  in (ii), we can precisely estimate the size of  $K^B(z)$  on  $\mathcal{U}_\alpha$ . The region  $\mathcal{U}_\alpha$  reminds us of the *admissible approach regions* considered in [33],[34],[27],[28],[1] etc. (b) The limit of  $K^B(z) \cdot r(z)^{|Q|+|\frac{1}{m}|_P+1}$  as  $z \rightarrow z^0$  on each  $\mathcal{U}_\alpha$  is not determined uniquely but depends on the angular variables  $t$ . Note that this boundary limit is uniquely determined on any nontangential cone. (c) In view of (1.4) the polar coordinates  $(t, r)$  is necessary to understand the asymptotic formula of  $K^B(z)$  at  $z^0$ . This fact may be interpreted that the Bergman kernel has a singularity of irregular singular type at a weakly pseudoconvex point. The degeneration from the strong pseudoconvexity to the weak pseudoconvexity corresponds to the process of confluence from the regular singularity to the irregular singularity ([30],[29]).

In more detail we investigate the structure of singularities of the Bergman kernel of  $\mathcal{E}_m$ . The singularities of  $\Phi^B(t)$  at  $\overline{\Delta} \setminus \Delta$ .  $\Phi^B(t)$  can also be expressed in a similar form to (4) by introducing new polar coordinates on the simplex  $\Delta$ . Through the finite recursive procedure of this type we can completely understand the structure of the singularities of  $K^B(z)$ .

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